System of equations. Application to Yang-Mills-Higgs monopole

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Outline

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2. Yang-Mills-Higgs monopole
3. The `Param_elliptic` class

INTRODUCTION

Type of problems

We have to solve a set of k partial differential equations, coupled :

$$H_i f_i = S_i \left(f_1, f_2 .... f_k\right) \quad \forall 0 \leq i < k$$

where H_i are differential operators (typically second order...)

Iteration technique

- Give an initial guess for the f_i.
- Computes the sources.
- Invert the operators H_i.
- If the relative change in the f_i is small stop, else compute the new sources and loop.

A few questions

Choice of H_i and f_i

- The choice of operators and functions can greatly influence the stability and convergence of the code.
- Typically, it is best if S_i contains only quadratic terms (or even higher order) in f_k.

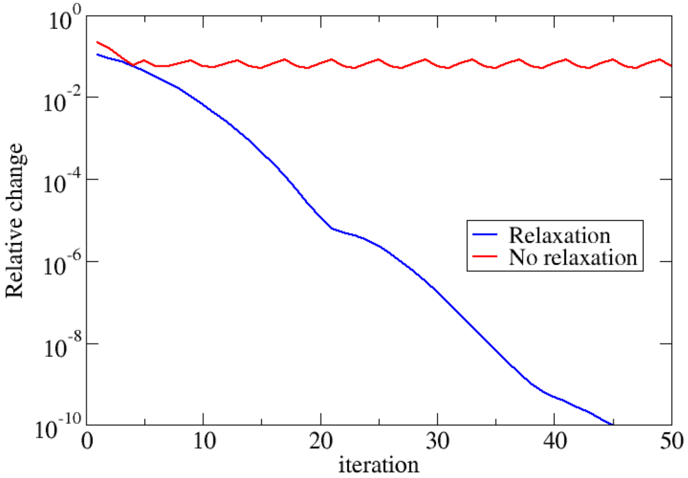
Relaxation

- If we replace simple f_i par $H_i^{-1}\left[S_i\right]$ the code usually diverges.
- We slow the change from step to step by using relaxation like :

$$f_i^{\text{new}} = \lambda H_i^{-1}\left[S_i\right] + (1-\lambda)\, f_i^{\text{old}}$$

- Typical values : $\lambda \approx 0.5$.

Influence of relaxation



H_i : simple cases

In LORENE a lot of choices are implemented for the operators H_i.

Members of `Scalar` :

- First order : primitive `primr`.
- Standard Poisson : `poisson` and `poisson_tau`.
- Poisson with inner boundary conditions : `poisson_dirichlet`, `poisson_neumann`, `poisson_dir_neum`.
- Angular part : `poisson_angu`.

The vectorial counterpart do also exist, members of `Vector`.
However, we may want to change H_i and f_i from domains to domains : need a more general solver...

YANG-MILLS-HIGGS MONOPOLE

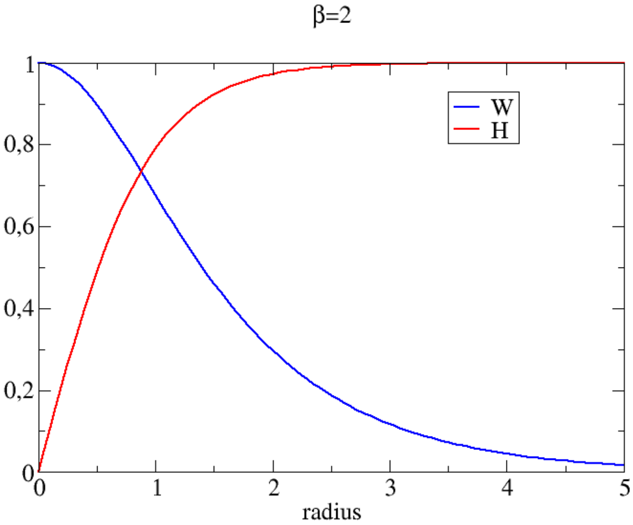
The equations

Monopoles are stationary, localized solutions of finite energy, in certain theories. Such solutions, in Yang-Mills-Higgs, assuming spherical symmetry, are solutions of a set of two equations :

$$\begin{aligned} W'' &= \frac{W\left(W^2-1\right)}{r^2} + WH^2 \\ H'' + \frac{2}{r}H' &= 2\frac{W^2H}{r^2} + \frac{\beta^2}{2}H\left(H^2-1\right) \end{aligned}$$

where W describe the gauge field, H the Higgs field, and β is a parameter giving the ratio of the masses.

Typical solutions



Asymptotic behavior

Near the origin :

- $W = 1 - ar^2 + \mathcal{O}\left(r^4\right)$
- $H = br + \mathcal{O}\left(r^3\right)$

At infinity

- W goes to zero exponentially.
- H goes to 1 exponentially.

Why not use Poisson operators ?

Suppose we write the system in the following form :

$$\begin{aligned} \Delta W &= S_W(W,H) \\ \Delta H &= S_H(W,H) \end{aligned}$$

- If we do not start from THE solution, after the first step, homogeneous solutions of the Laplacian do appear.
- Those homogeneous solutions are in $1/r$ and do not decay fast enough for the sources.
- the code crashed very quickly.

One needs to maintain exponential convergence throughout the iteration.

Equations near infinity

In the CED, one will work with W and $h = H - 1$.
One can make Helmholtz operators appear :

$$\begin{aligned} \Delta_{l=0} W - W &= hW\left(h+2\right) + \frac{W\left(W^2-1\right)}{r^2} + 2\frac{W'}{r} \\ \Delta_{l=0} h - \beta^2 h &= 2\frac{W^2\left(h+1\right)}{r^2} + \frac{\beta^2}{2} h^2 \left(h+3\right) \end{aligned}$$

Homogeneous solutions are exponentials $\Longrightarrow$ exponential convergence is maintained during the iteration.

Near the origin

- One can see that $W = 0$ everywhere is solution of the system of equations.
- One should prevent the code to converge to $W = 0$.
- Near the origin, one can use :

$$w = \frac{W - 1}{r}$$

Given that : $W = 1 - ar^2 + \mathcal{O}\left(r^4\right)$

- w is regular at the origin.
- w is odd near the origin.
- It forces $W\left(r = 0\right) = 1$.

Equations near the origin

In the nucleus, one will work with w and H.
The equations can be put in the following form :

$$\begin{aligned}\Delta_{l=1} w \equiv w'' + \frac{2}{r} w' - 2\frac{w}{r^2} &= w^3 + 3\frac{w^2}{r} + (1 + rw)\frac{H^2}{r} \\ \Delta_{l=1} H \equiv H'' + \frac{2}{r} H' - 2\frac{H}{r^2} &= 2H\left(w^2 + 2\frac{w}{r}\right) + \frac{\beta^2}{2} H\left(H^2 - 1\right).\end{aligned}$$

Equations in the shells

Neither of the functions are singular.
We can use W and H and write the equations like :

$$\begin{aligned}\Delta_{l=0}W - W &= W\left(H^2-1\right) + \frac{W\left(W^2-1\right)}{r^2} + 2\frac{W'}{r}\\ \Delta_{l=0}H - \beta^2 H &= 2\frac{W^2 H}{r^2} + \frac{\beta^2}{2}H\left(H^2-3\right)\end{aligned}$$

THE Param_elliptic CLASS

So what is needed ?

We need to solve equations of the type : $H_d f_d = S_d$ where d denotes the domain

- H_d can be different operators in each domain.
- f_d are auxiliary variables, related to the "real" one F.
- The auxiliary variables can be different from domain to domain.
- The continuous function is F.

Use the `Param_elliptic` object

- constructor : `Param_elliptic(const Scalar & so)`
- various functions to initialize the `Param_elliptic`, setting both
  - the H_d
  - the variable changes.
- Solve the equation by calling `Scalar::sol_elliptic (Param_elliptic)`
- It returns a `Scalar` containing f_d.

Setting the variable change

For now the auxiliary variables should be of the type :

$$W = F(r, \theta, \varphi) + G(r) w$$

F and G are given in every domain

- By default $F = 0$ and $G = 1$
- F is changed by `Param_elliptic::set_variable_F (const Scalar &)`
- G is changed by `Param_elliptic::set_variable_G (const Scalar &)`.
- Be careful : the code does not check that G is only a function of r

Monopole variable change

For W

- In the nucleus : $W = 1 + rw$
- W elsewhere.

For H

- In the CED : $H = 1 + h$
- H elsewhere.

They are of the right type

Changing the operators

Use the provided member functions of Param_elliptic like :

- `set_helmholtz_minus (int zone, double m, Scalar &so)`

  sets the operator to $\Delta - m^2$ in the domain zone.

- `inc_l_quant (int zone)`

  increases l in the domain zone, the operator being of the type :

$$f'' + \frac{2}{r} f' - \frac{l\,(l+1)}{r^2} f$$