An introduction to polynomial interpolation

Eric Gourgoulhon

Laboratoire de l'Univers et de ses Théories (LUTH)
CNRS / Observatoire de Paris
F-92195 Meudon, France

eric.gourgoulhon@obspm.fr

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Plan

Outline

Introduction

Basic idea: approximate functions $\mathbb{R} \to \mathbb{R}$ by polynomials

Polynomials are the only functions that a computer can evaluate exactely.

Two types of numerical methods based on polynomial approximations:

- **spectral methods:** high order polynomials on a single domain (or a few domains)
- **finite elements**: low order polynomials on many domains

Introduction

Basic idea: approximate functions $\mathbb{R} \to \mathbb{R}$ by polynomials

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- **spectral methods**: high order polynomials on a single domain (or a few domains)
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Framework of this lecture

We consider real-valued functions on the compact interval $[-1, 1]$:

$$f : [-1, 1] \longrightarrow \mathbb{R}$$

We denote

- by $\mathbb{P}$ the set all real-valued polynomials on $[-1, 1]$:

$$\forall p \in \mathbb{P}, \ \forall x \in [-1, 1], \ p(x) = \sum_{i=0}^{n} a_i \, x^i$$

- by $\mathbb{P}_N$ (where N is a positive integer), the subset of polynomials of degree at most N.

Is it a good idea to approximate functions by polynomials ?

For **continuous functions**, the answer is **yes**:

Theorem (Weierstrass, 1885)

$\mathbb{P}$ is a dense subspace of the space $C^0([-1,1])$ of all continuous functions on $[-1,1]$, equiped with the uniform norm $\|.\|_\infty$.[a]

[a]This is a particular case of the *Stone-Weierstrass theorem*

The uniform norm or maximum norm is defined by $\|f\|_\infty = \max_{x\in[-1,1]} |f(x)|$

Other phrasings:

For any continuous function on $[-1,1]$, f, and any $\epsilon > 0$, there exists a polynomial $p \in \mathbb{P}$ such that $\|f - p\|_\infty < \epsilon$.

For any continuous function on $[-1,1]$, f, there exists a sequence of polynomials $(p_n)_{n\in\mathbb{N}}$ which converges uniformly towards f: $\lim_{n\to\infty} \|f - p_n\|_\infty = 0$.

Best approximation polynomial

For a given continuous function: $f \in C^0([-1,1])$, a best approximation polynomial of degree N is a polynomial $p_N^*(f) \in \mathbb{P}_N$ such that

$$\|f - p_N^*(f)\|_\infty = \min\left\{\|f - p\|_\infty,\ p \in \mathbb{P}_N\right\}$$

Chebyshev's alternant theorem (or equioscillation theorem)

For any $f \in C^0([-1,1])$ and $N \geq 0$, the best approximation polynomial $p_N^*(f)$ exists and is unique. Moreover, there exists $N+2$ points x_0, x_1, ... x_{N+1} in [-1,1] such that

$$f(x_i) - p_N^*(f)(x_i) = (-1)^i \,\|f - p_N^*(f)\|_\infty, \qquad 0 \leq i \leq N+1$$

or $$f(x_i) - p_N^*(f)(x_i) = (-1)^{i+1} \,\|f - p_N^*(f)\|_\infty, \qquad 0 \leq i \leq N+1$$

Corollary: $p_N^*(f)$ interpolates f in $N+1$ points.

Illustration of Chebyshev's alternant theorem

$N = 1$

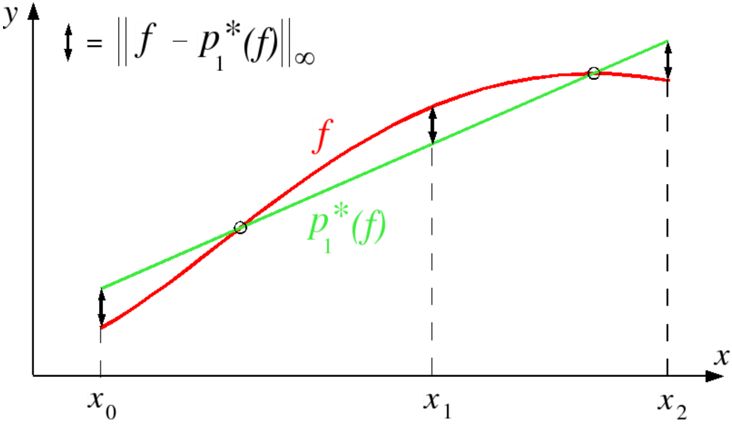
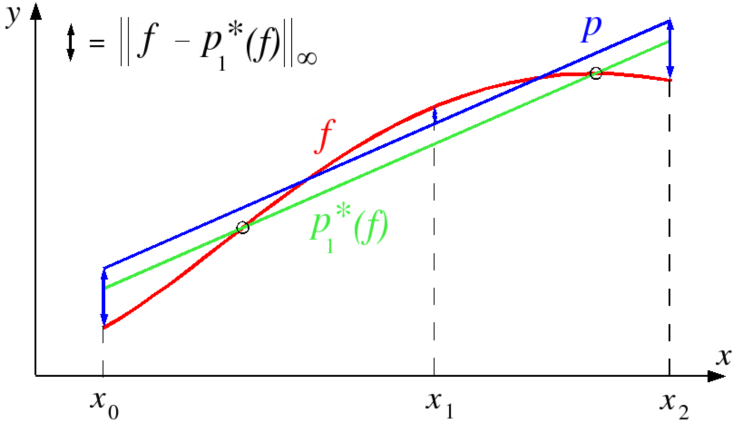


Illustration of Chebyshev's alternant theorem

$N = 1$



Outline

Interpolation on an arbitrary grid

Definition: given an integer $N \geq 1$, a grid is a set of $N+1$ points $X = (x_i)_{0\leq i\leq N}$ in [-1,1] such that $-1 \leq x_0 < x_1 < \cdots < x_N \leq 1$. The $N+1$ points $(x_i)_{0\leq i\leq N}$ are called the nodes of the grid.

Theorem

Given a function $f \in C^0([-1,1])$ and a grid of $N+1$ nodes, $X = (x_i)_{0\leq i\leq N}$, there exist a unique polynomial of degree N, $I_N^X f$, such that

$$I_N^X f(x_i) = f(x_i), \qquad 0 \leq i \leq N$$

$I_N^X f$ is called the interpolant (or the interpolating polynomial) of f through the grid X.

Lagrange form of the interpolant

The interpolant $I_N^X f$ can be expressed in the *Lagrange form:*

$$I_N^X f(x) = \sum_{i=0}^{N} f(x_i)\, \ell_i^X(x),$$

where $\ell_i^X(x)$ is the i-th Lagrange cardinal polynomial associated with the grid X:

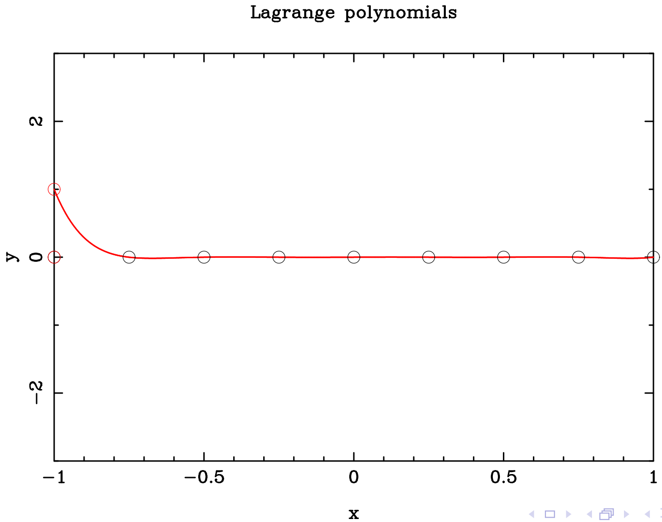
$$\boxed{\ell_i^X(x) := \prod_{\substack{j=0 \\ j\neq i}}^{N} \frac{x - x_j}{x_i - x_j}, \qquad 0 \leq i \leq N}$$

The Lagrange cardinal polynomials are such that

$$\ell_i^X(x_j) = \delta_{ij}, \qquad 0 \leq i, j \leq N$$

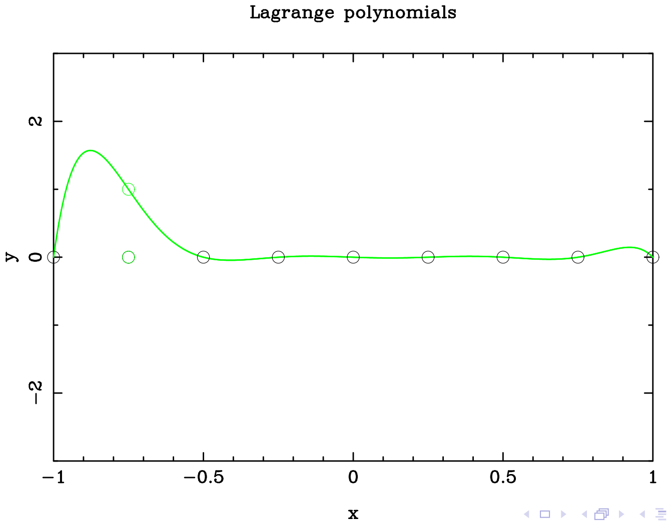
Examples of Lagrange polynomials

Uniform grid $N = 8$ $\qquad \ell_0^X(x)$



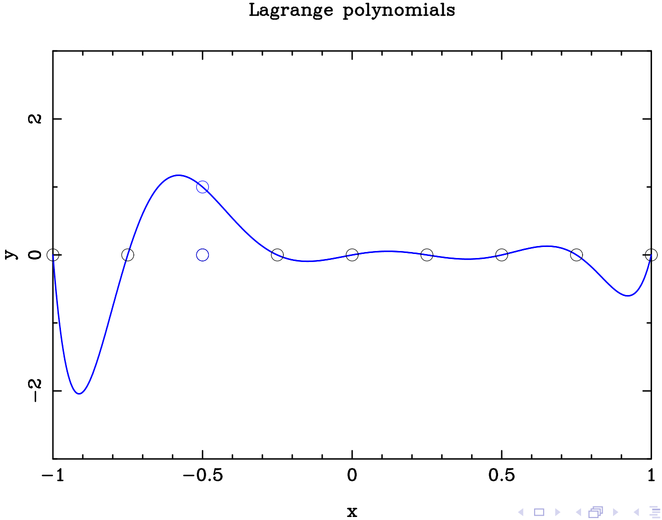
Examples of Lagrange polynomials

Uniform grid $N = 8$ $\ell_1^X(x)$



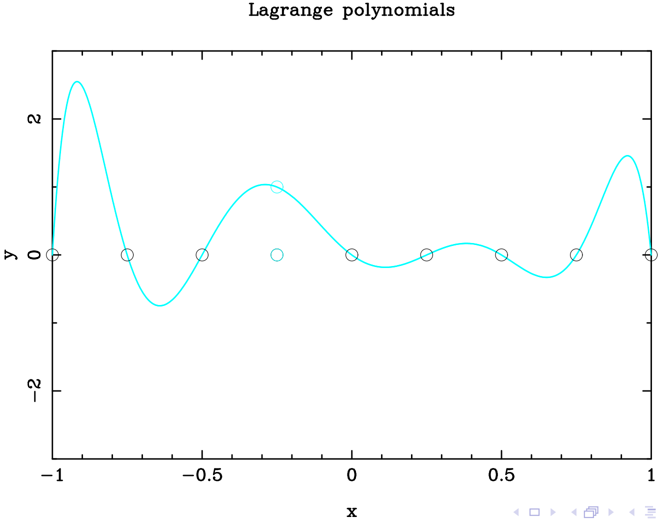
Examples of Lagrange polynomials

Uniform grid $N = 8$ $\ell_2^X(x)$



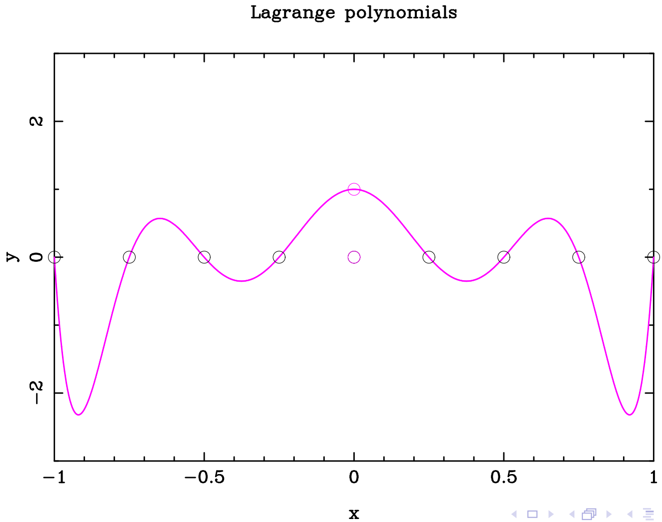
Examples of Lagrange polynomials

Uniform grid $N = 8$ $\ell_3^X(x)$



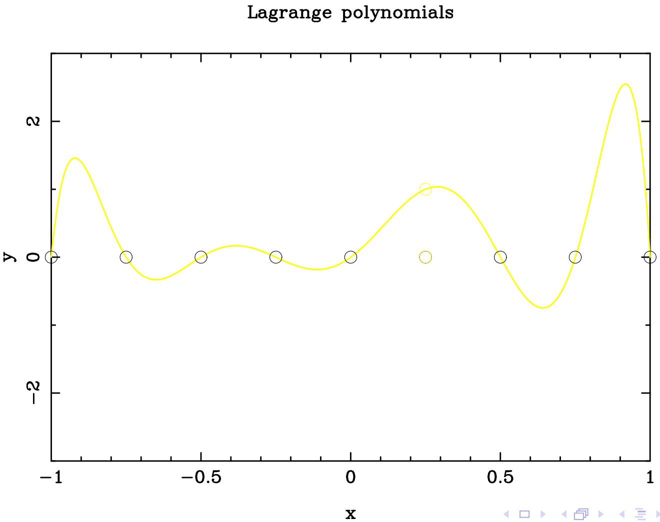
Examples of Lagrange polynomials

Uniform grid $N = 8$ $\ell_4^X(x)$



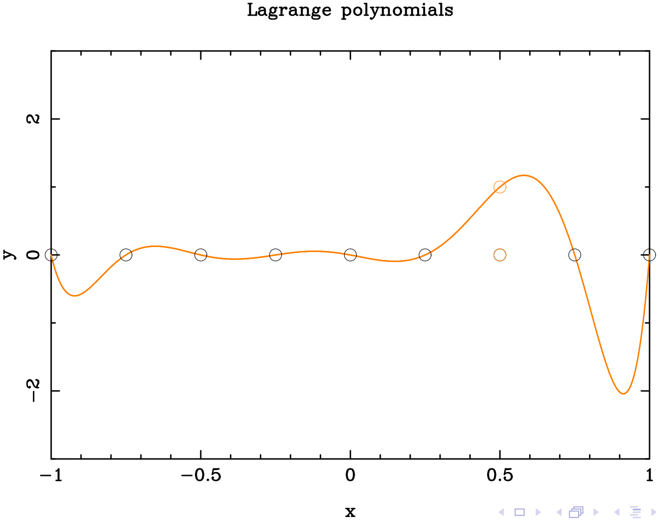
Examples of Lagrange polynomials

Uniform grid $N = 8$ $\ell_5^X(x)$



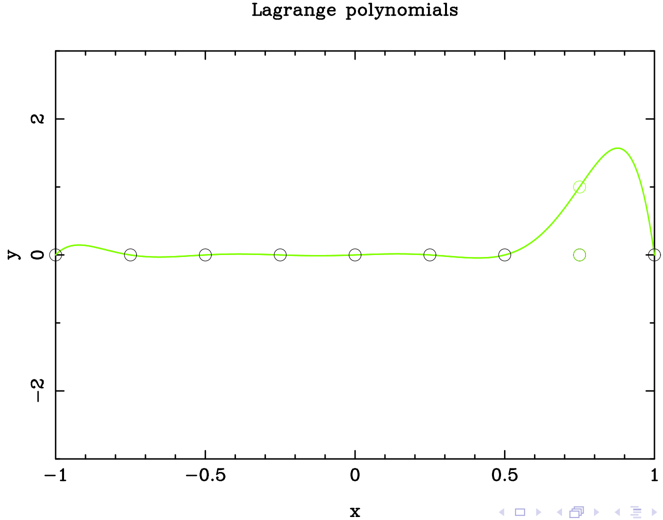
Examples of Lagrange polynomials

Uniform grid $N = 8$ $\ell_6^X(x)$



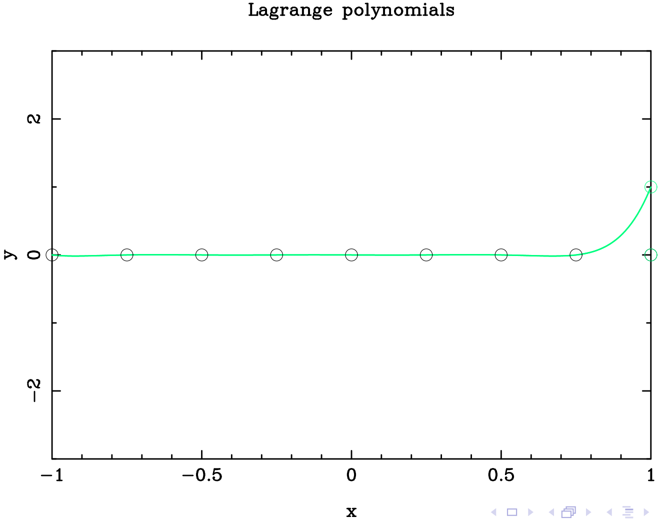
Examples of Lagrange polynomials

Uniform grid $N = 8$ $\qquad \ell_7^X(x)$



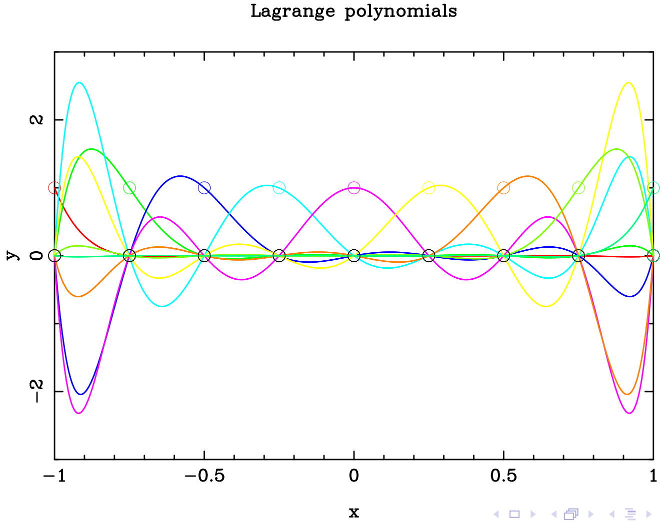
Examples of Lagrange polynomials

Uniform grid $N = 8$ $\ell_8^X(x)$



Examples of Lagrange polynomials

Uniform grid $N = 8$



Interpolation error with respect to the best approximation error

Let $N \in \mathbb{N}$, $X = (x_i)_{0 \le i \le N}$ a grid of $N+1$ nodes and $f \in C^0([-1,1])$.

Let us consider the interpolant $I_N^X f$ of f through the grid X.

The best approximation polynomial $p_N^*(f)$ is also an interpolant of f at $N+1$ nodes (in general different from X) reminder

How does the error $\|f - I_N^X f\|_\infty$ behave with respect to the smallest possible error $\|f - p_N^*(f)\|_\infty$?

The answer is given by the formula:

$$\boxed{\|f - I_N^X f\|_\infty \le (1 + \Lambda_N(X))\, \|f - p_N^*(f)\|_\infty}$$

where $\Lambda_N(X)$ is the Lebesgue constant relative to the grid X:

$$\Lambda_N(X) := \max_{x\in[-1,1]} \sum_{i=0}^{N} \left|\ell_i^X(x)\right|$$

Lebesgue constant

The Lebesgue constant contains all the information on the effects of the choice of X on $\|f - I_N^X f\|_\infty$.

Theorem (Erdős, 1961)

For any choice of the grid X, there exists a constant $C > 0$ such that

$$\Lambda_N(X) > \frac{2}{\pi} \ln(N+1) - C$$

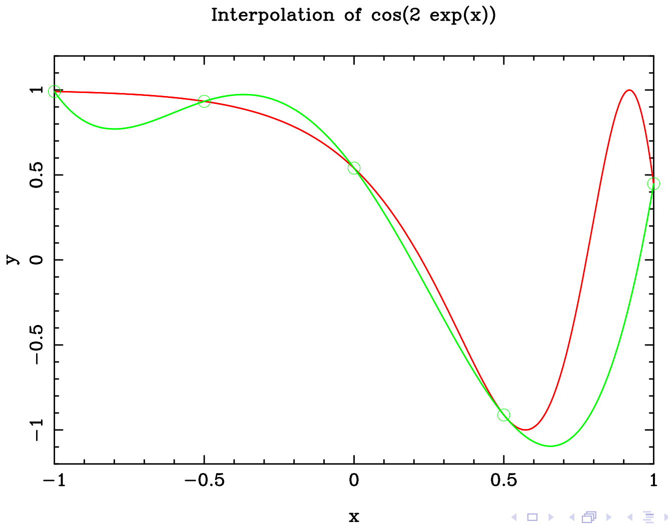
Corollary: $\Lambda_N(X) \to \infty$ as $N \to \infty$

In particular, for a **uniform** grid, $\Lambda_N(X) \sim \dfrac{2^{N+1}}{e N \ln N}$ as $N \to \infty$!

This means that for any choice of type of sampling of $[-1, 1]$, there exists a continuous function $f \in C^0([-1, 1])$ such that $I_N^X f$ does not convergence uniformly towards f.

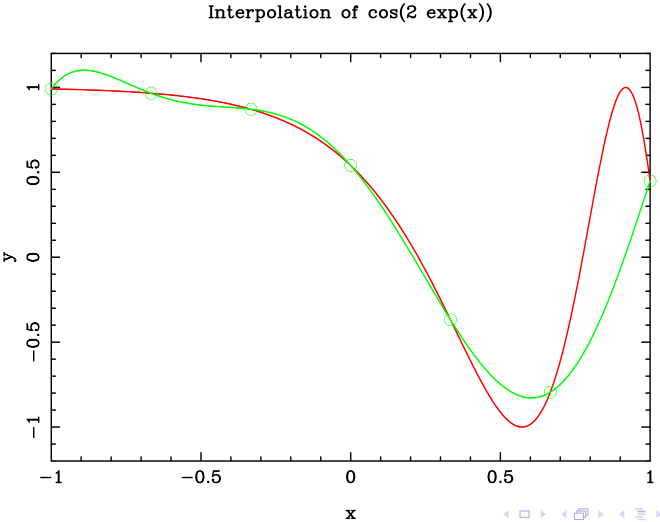
Example: uniform interpolation of a "gentle" function

$f(x) = \cos(2\exp(x))$ uniform grid $N = 4$: $\|f - I_4^X f\|_\infty \simeq 1.40$



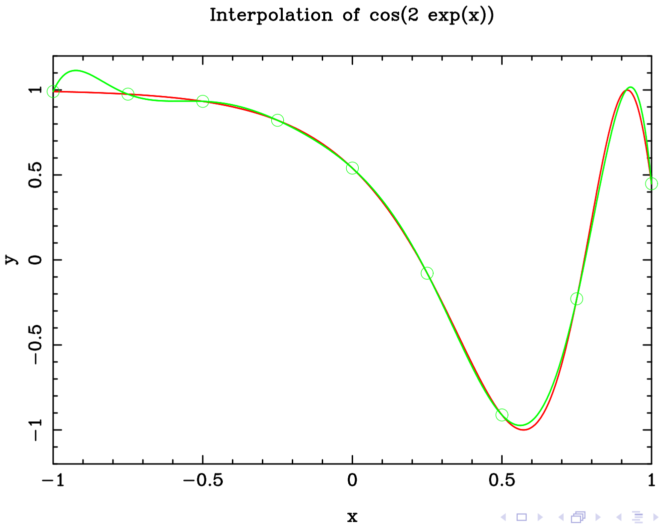
Example: uniform interpolation of a "gentle" function

$f(x) = \cos(2\exp(x))$ uniform grid $N = 6$: $\|f - I_6^X f\|_\infty \simeq 1.05$



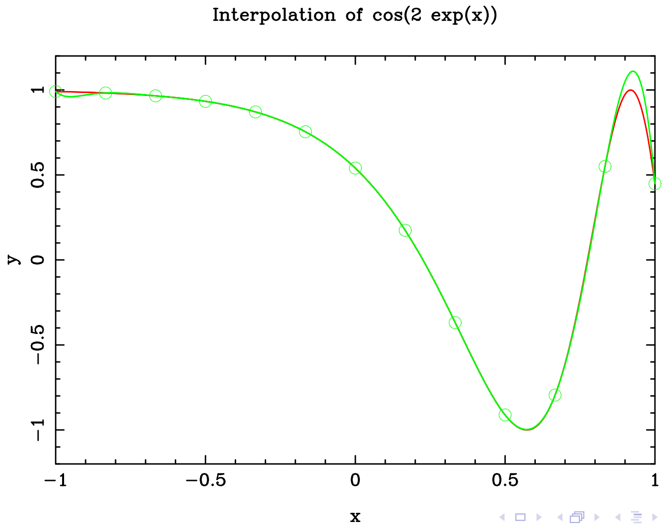
Example: uniform interpolation of a “gentle” function

$f(x) = \cos(2\exp(x))$ uniform grid $N = 8$: $\|f - I_8^X f\|_\infty \simeq 0.13$



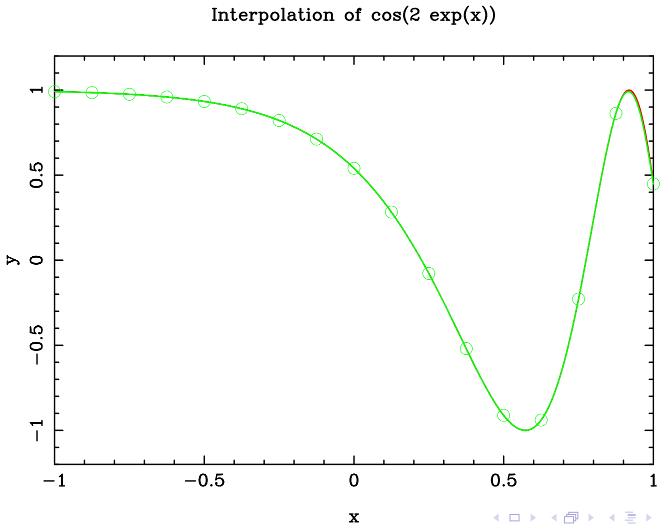
Example: uniform interpolation of a "gentle" function

$f(x) = \cos(2\exp(x))$ uniform grid $N = 12$: $\|f - I_{12}^X f\|_\infty \simeq 0.13$



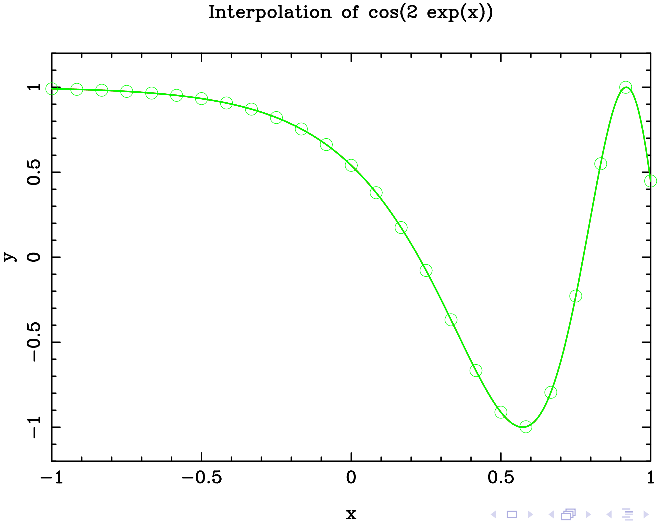
Example: uniform interpolation of a “gentle” function

$f(x) = \cos(2\exp(x))$ uniform grid $N = 16$: $\|f - I_{16}^X f\|_\infty \simeq 0.025$



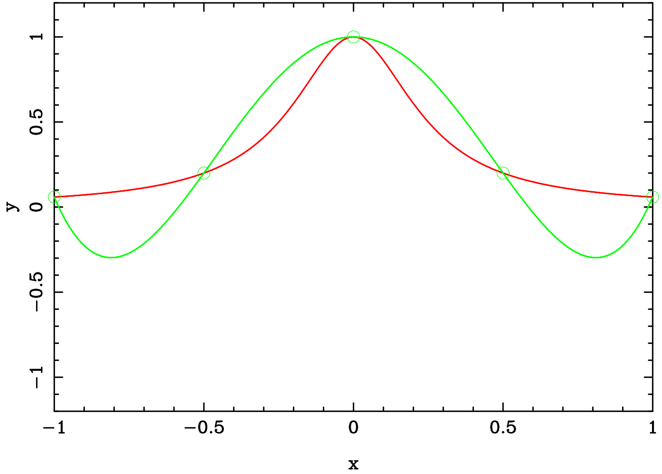
Example: uniform interpolation of a "gentle" function

$f(x) = \cos(2\exp(x))$ uniform grid $N = 24$: $\|f - I^X_{24} f\|_\infty \simeq 4.6\,10^{-4}$



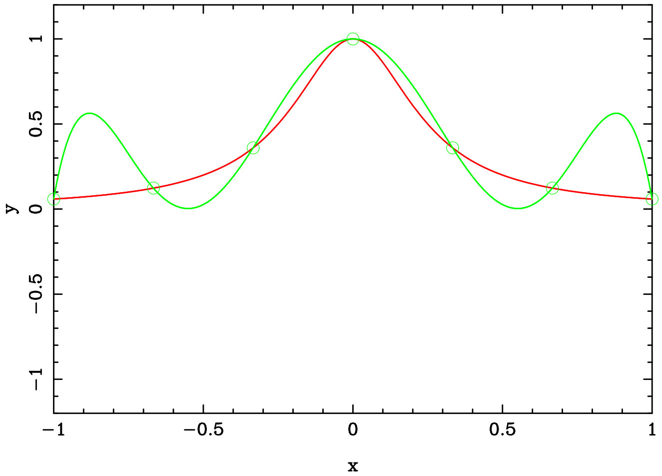
Runge phenomenon

$$f(x) = \frac{1}{1+16x^2}$$ uniform grid $N = 4$: $\|f - I_4^X f\|_\infty \simeq 0.39$



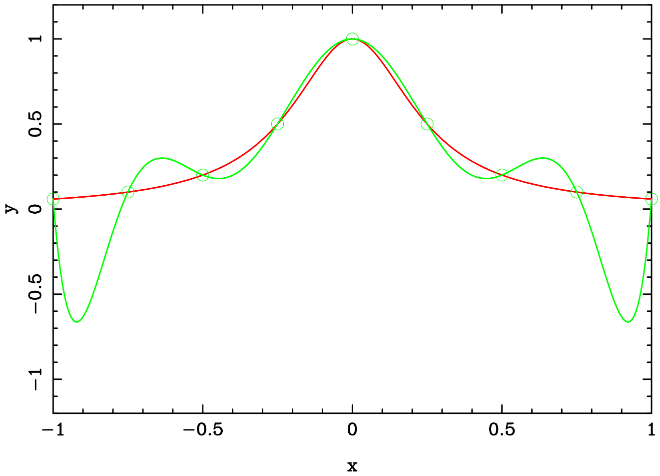
Runge phenomenon

$f(x) = \frac{1}{1+16x^2}$ uniform grid $N = 6$: $\|f - I_6^X f\|_\infty \simeq 0.49$



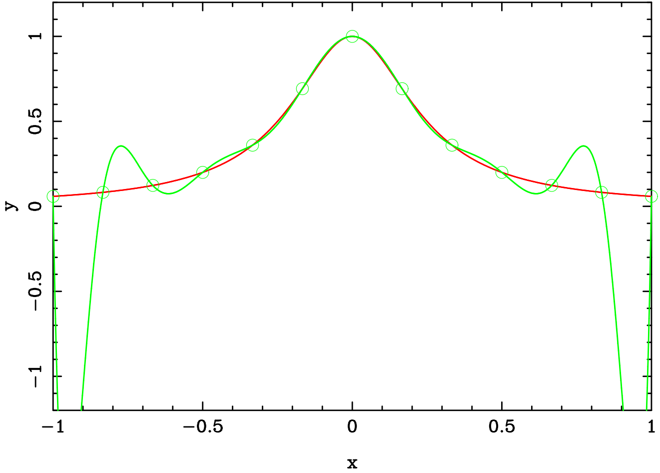
Runge phenomenon

$$f(x) = \frac{1}{1+16x^2}$$ uniform grid $N = 8$: $\|f - I_8^X f\|_\infty \simeq 0.73$



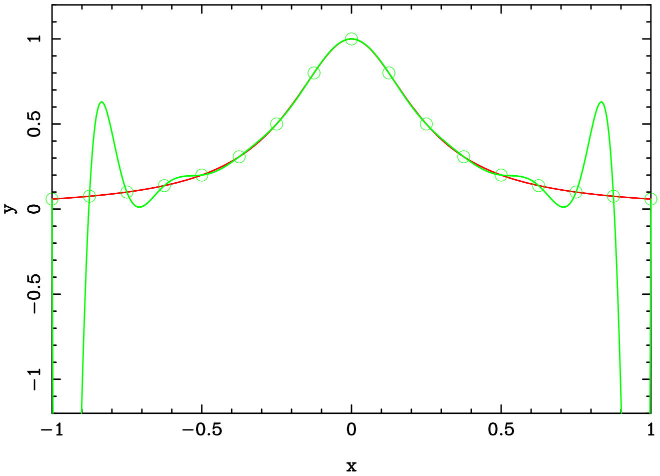
Runge phenomenon

$f(x) = \frac{1}{1+16x^2}$ uniform grid $N = 12$: $\|f - I_{12}^X f\|_\infty \simeq 1.97$



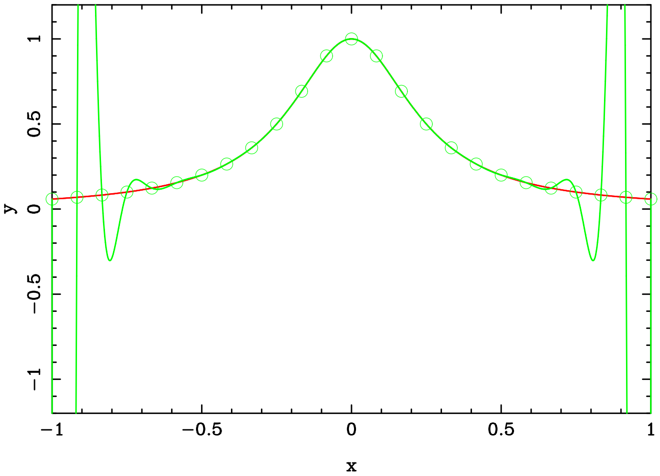
Runge phenomenon

$f(x) = \dfrac{1}{1+16x^2}$ uniform grid $N = 16$: $\|f - I_{16}^X f\|_\infty \simeq 5.9$



Runge phenomenon

$$f(x) = \frac{1}{1+16x^2}$$ uniform grid $N = 24$: $\|f - I^X_{24} f\|_\infty \simeq 62$



Evaluation of the interpolation error

Let us assume that the function f is sufficiently smooth to have derivatives at least up to the order $N+1$, with $f^{(N+1)}$ continuous, i.e. $f \in C^{N+1}([-1,1])$.

Theorem (Cauchy)

If $f \in C^{N+1}([-1,1])$, then for any grid X of $N+1$ nodes, and for any $x \in [-1,1]$, the interpolation error at x is

$$f(x) - I_N^X(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!}\,\omega_{N+1}^X(x) \qquad (1)$$

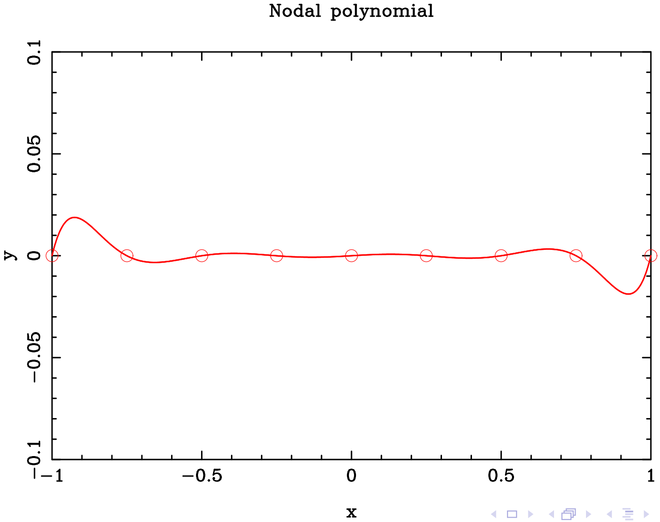
where $\xi = \xi(x) \in [-1,1]$ and $\omega_{N+1}^X(x)$ is the nodal polynomial associated with the grid X.

Definition: The nodal polynomial associated with the grid X is the unique polynomial of degree $N+1$ and leading coefficient 1 whose zeros are the $N+1$ nodes of X:

$$\boxed{\omega_{N+1}^X(x) := \prod_{i=0}^{N}(x - x_i)}$$

Example of nodal polynomial

Uniform grid $N = 8$



Minimizing the interpolation error by the choice of grid

In Eq. (1), we have no control on $f^{(N+1)}$, which can be large.
For example, for $f(x) = 1/(1+\alpha^2 x^2)$, $\|f^{(N+1)}\|_\infty = (N+1)!\,\alpha^{N+1}$.

Idea: choose the grid X so that $\omega^X_{N+1}(x)$ is small, i.e. $\|\omega^X_{N+1}\|_\infty$ is small.

Notice: $\omega^X_{N+1}(x)$ has leading coefficient 1: $\omega^X_{N+1}(x) = x^{N+1} + \sum_{i=0}^{N} a_i\, x^i$.

Theorem (Chebyshev)

Among all the polynomials of degree $N+1$ and leading coefficient 1, the unique polynomial which has the smallest uniform norm on $[-1,1]$ is the $(N+1)$-th **Chebyshev polynomial** *divided by 2^N: $T_{N+1}(x)/2^N$.*

Since $\|T_{N+1}\|_\infty = 1$, we conclude that if we choose the grid nodes $(x_i)_{0\le i\le N}$ to be the $N+1$ zeros of the Chebyshev polynomial T_{N+1}, we have

$$\|\omega^X_{N+1}\|_\infty = \frac{1}{2^N}$$

and this is the smallest possible value.

Chebyshev-Gauss grid

The grid $X = (x_i)_{0\leq i\leq N}$ such that the x_i's are the $N+1$ zeros of the Chebyshev polynomial of degree $N+1$ is called the Chebyshev-Gauss (CG) grid.
It has much better interpolation properties than the uniform grid considered so far.
In particular, from Eq. (1), for any function $f \in C^{N+1}([-1,1])$,

$$\left\| f - I_N^{\mathrm{CG}} f \right\|_\infty \leq \frac{1}{2^N (N+1)!} \left\| f^{(N+1)} \right\|_\infty$$

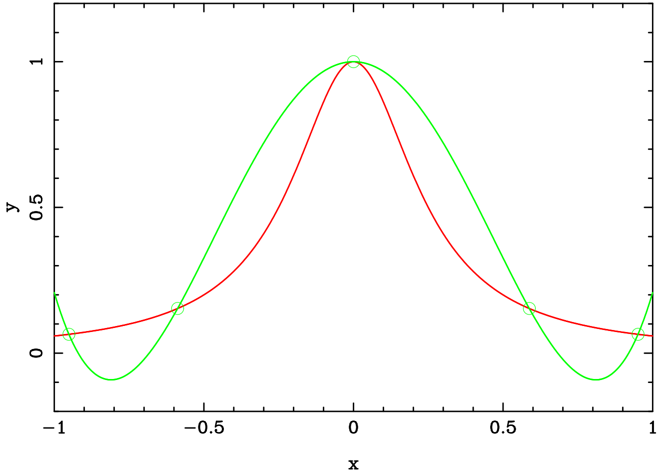
If $f^{(N+1)}$ is uniformly bounded, the convergence of the interpolant $I_N^{\mathrm{CG}} f$ towards f when $N \to \infty$ is then extremely fast.
Also the Lebesgue constant associated with the Chebyshev-Gauss grid is small:

$$\Lambda_N(CG) \sim \frac{2}{\pi} \ln(N+1) \quad \text{as} \quad N \to \infty$$

This is much better than uniform grids and close to the optimal value reminder

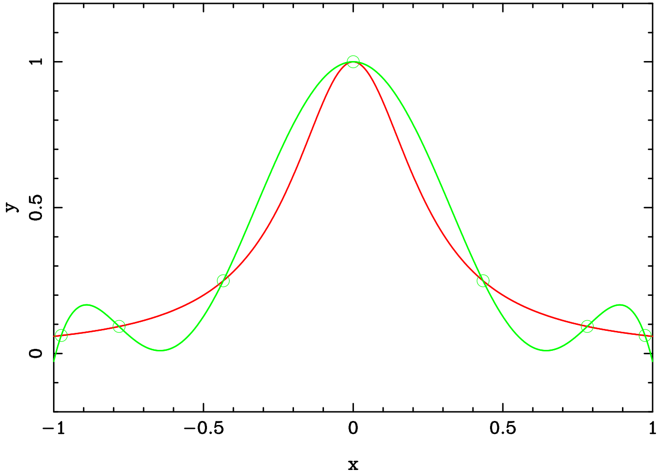
Example: Chebyshev-Gauss interpolation of $f(x) = \frac{1}{1+16x^2}$

$f(x) = \dfrac{1}{1+16x^2}$ CG grid $N = 4$: $\|f - I_4^{\text{CG}} f\|_\infty \simeq 0.31$



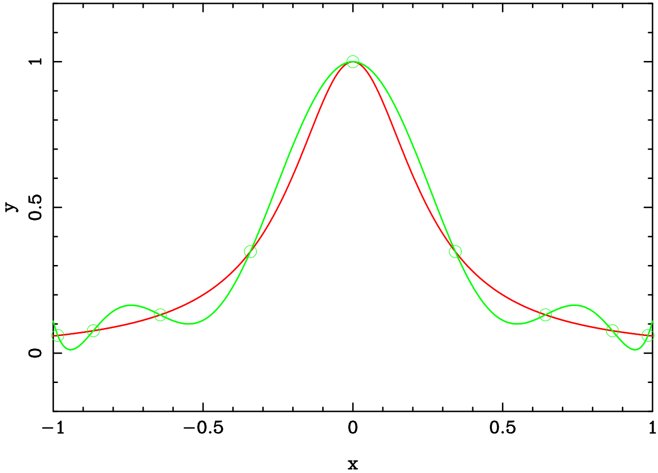
Example: Chebyshev-Gauss interpolation of $f(x) = \frac{1}{1+16x^2}$

$f(x) = \dfrac{1}{1+16x^2}$ CG grid $N = 6$: $\|f - I_6^{\mathrm{CG}} f\|_\infty \simeq 0.18$



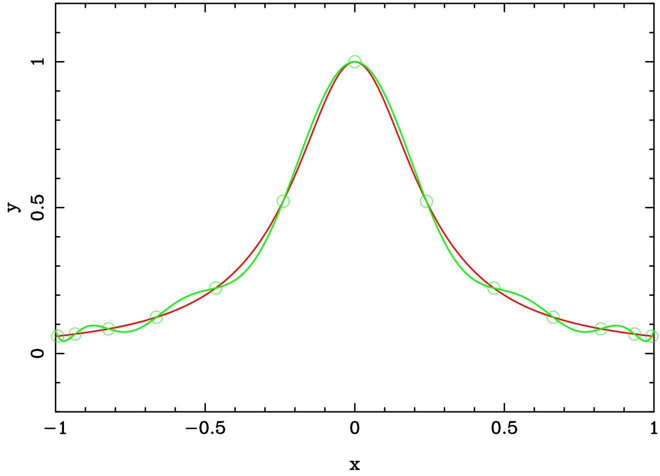
Example: Chebyshev-Gauss interpolation of $f(x) = \frac{1}{1+16x^2}$

$f(x) = \dfrac{1}{1+16x^2}$ CG grid $N = 8$: $\|f - I_8^{\rm CG} f\|_\infty \simeq 0.10$



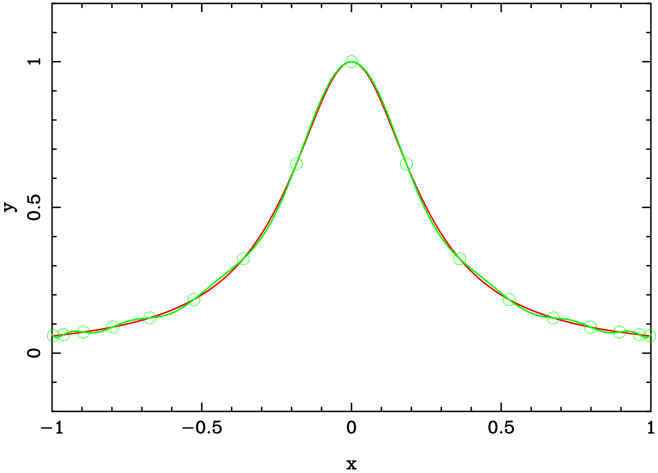
Example: Chebyshev-Gauss interpolation of $f(x) = \frac{1}{1+16x^2}$

$f(x) = \dfrac{1}{1 + 16x^2}$ CG grid $N = 12$: $\|f - I_{12}^{CG} f\|_\infty \simeq 3.8\ 10^{-2}$



Example: Chebyshev-Gauss interpolation of $f(x) = \frac{1}{1+16x^2}$

$f(x) = \dfrac{1}{1+16x^2}$ CG grid $N = 16$: $\|f - I_{16}^{\mathrm{CG}} f\|_\infty \simeq 1.5\ 10^{-2}$

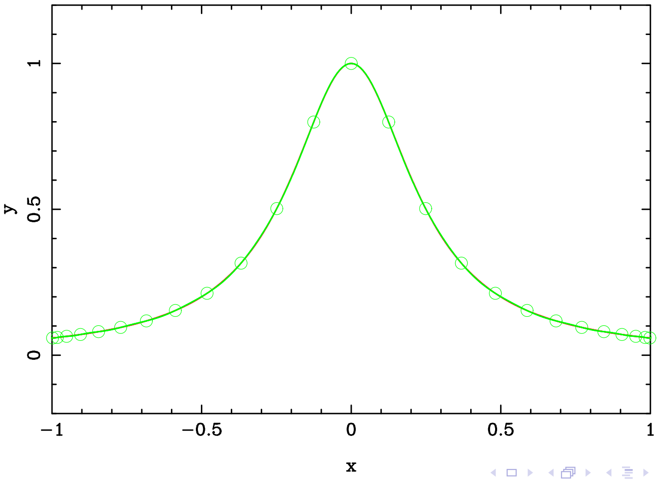


Example: Chebyshev-Gauss interpolation of $f(x) = \frac{1}{1+16x^2}$

$$f(x) = \frac{1}{1+16x^2}$$

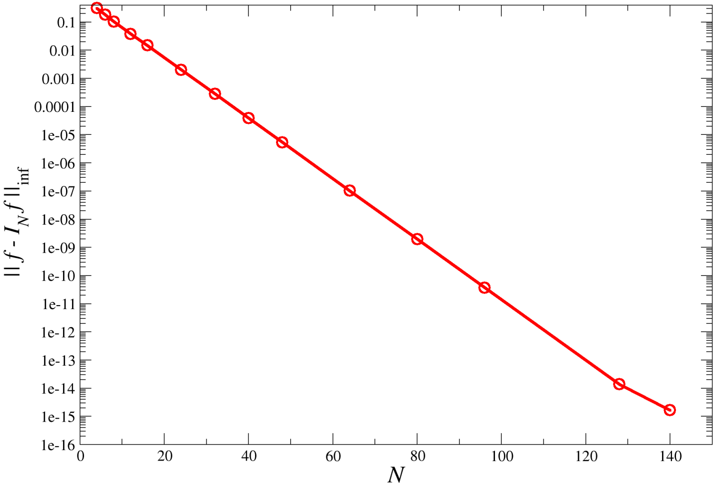
CG grid $N = 24$: $\|f - I_{24}^{\mathrm{CG}} f\|_\infty \simeq 2.0\ 10^{-3}$

no Runge phenomenon !



Example: Chebyshev-Gauss interpolation of $f(x) = \frac{1}{1+16x^2}$

Variation of the interpolation error as N increases



Chebyshev polynomials = orthogonal polynomials

The Chebyshev polynomials, the zeros of which provide the Chebyshev-Gauss nodes, constitute a family of **orthogonal polynomials**, and the Chebyshev-Gauss nodes are associated to **Gauss quadratures**.

Outline

Hilbert space $L^2_w(-1,1)$

Framework: Let us consider the functional space

$$L^2_w(-1,1) = \left\{ f : (-1,1) \to \mathbb{R}, \ \int_{-1}^{1} f(x)^2 \, w(x) \, dx < \infty \right\}$$

where $w: \ (-1,1) \to (0,\infty)$ is an integrable function, called the weight function.

$L^2_w(-1,1)$ is a **Hilbert space** for the scalar product

$$(f|g)_w := \int_{-1}^{1} f(x) \, g(x) \, w(x) \, dx$$

with the associated norm

$$\|f\|_w := (f|f)_w^{1/2}$$

Orthogonal polynomials

The set $\mathbb{P}$ of polynomials on $[-1, 1]$ is a subspace of $L^2_w(-1, 1)$.
A family of orthogonal polynomials is a set $(p_i)_{i\in\mathbb{N}}$ such that

- $p_i \in \mathbb{P}$
- $\deg p_i = i$
- $i \neq j \Rightarrow (p_i|p_j)_w = 0$

$(p_i)_{i\in\mathbb{N}}$ is then a basis of the vector space $\mathbb{P}$: $\mathbb{P} = \mathrm{span}\,\{p_i, i \in \mathbb{N}\}$

Theorem

A family of orthogonal polynomial $(p_i)_{i\in\mathbb{N}}$ *is a* **Hilbert basis** *of* $L^2_w(-1, 1)$ *:*

$$\forall f \in L^2_w(-1,1), \quad f = \sum_{i=0}^{\infty} \tilde{f}_i \, p_i \qquad \text{with } \tilde{f}_i := \frac{(f|p_i)_w}{\|p_i\|^2_w}.$$

The above infinite sum means $\lim_{N\to\infty} \left\| f - \sum_{i=0}^{N} \tilde{f}_i \, p_i \right\|_w = 0$

Jacobi polynomials

Jacobi polynomials are orthogonal polynomials with respect to the weight

$$w(x) = (1-x)^\alpha (1+x)^\beta$$

Subcases:

- Legendre polynomials $P_n(x)$: $\alpha = \beta = 0$, i.e. $w(x) = 1$
- Chebyshev polynomials $T_n(x)$: $\alpha = \beta = -\frac{1}{2}$, i.e. $w(x) = \frac{1}{\sqrt{1-x^2}}$

Jacobi polynomials are eigenfunctions of the singular[1] **Sturm-Liouville problem**

$$-\frac{d}{dx}\left[(1-x^2)\, w(x)\, \frac{du}{dx}\right] = \lambda\, w(x)\, u, \qquad x \in (-1,1)$$

[1] *singular* means that the coefficient in front of du/dx vanishes at the extremities of the interval $[-1,1]$

Legendre polynomials

$$w(x) = 1: \quad \int_{-1}^{1} P_i(x) P_j(x)\, dx = \frac{2}{2i+1}\delta_{ij}$$

$P_0(x) = 1$

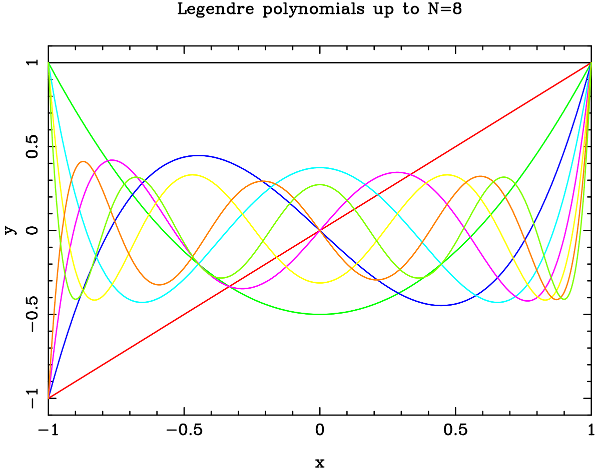
$P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1)$

$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$P_{i+1}(x) = \frac{2i+1}{i+1}\, x P_i(x) - \frac{i}{i+1} P_{i-1}(x)$



Chebyshev polynomials

$$w(x) = \frac{1}{\sqrt{1-x^2}}: \quad \int_{-1}^{1} T_i(x) T_j(x) \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}(1+\delta_{0i})\,\delta_{ij}$$

$T_0(x) = 1$

$T_1(x) = x$

$T_2(x) = 2x^2 - 1$

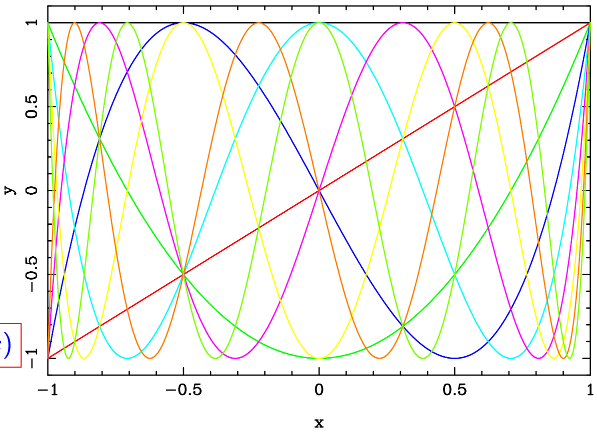
$T_3(x) = 4x^3 - 3x$

$T_4(x) = 8x^4 - 8x^2 + 1$

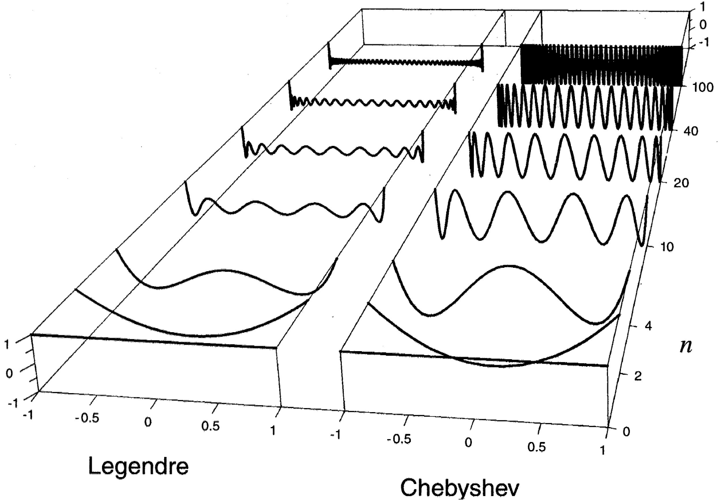
$$\cos(n\theta) = T_n(\cos\theta)$$

$$T_{i+1}(x) = 2xT_i(x) - T_{i-1}(x)$$

Chebyshev polynomials up to N=8



Legendre and Chebyshev compared



[from Fornberg (1998)]

Orthogonal projection on $\mathbb{P}_N$

Let us consider $f \in L^2_w(-1,1)$ and a family $(p_i)_{i\in\mathbb{N}}$ of orthogonal polynomials with respect to the weight w.
Since $(p_i)_{i\in\mathbb{N}}$ is a Hilbert basis of $L^2_w(-1,1)$ reminder
we have $f(x) = \sum_{i=0}^{\infty} \tilde{f}_i \, p_i(x)$ with $\tilde{f}_i := \dfrac{(f|p_i)_w}{\|p_i\|_w^2}$.
The truncated sum

$$\Pi^w_N f(x) := \sum_{i=0}^{N} \tilde{f}_i \, p_i(x)$$

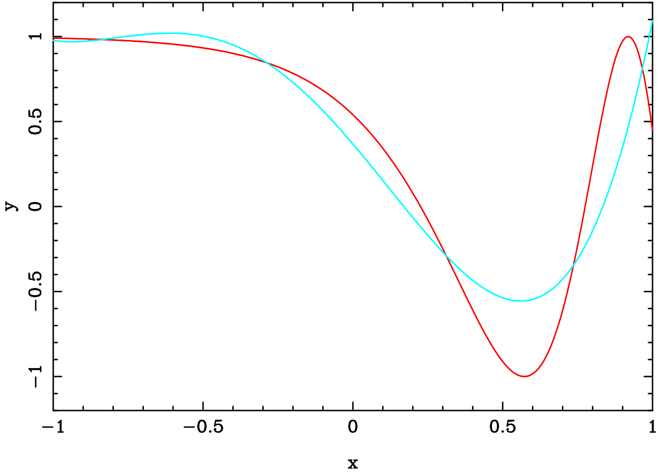
is a polynomial of degree N: it is the orthogonal projection of f onto the finite dimensional subspace $\mathbb{P}_N$ with respect to the scalar product $(.|.)_w$.
We have

$$\lim_{N\to\infty} \|f - \Pi^w_N f\|_w = 0$$

Hence $\Pi^w_N f$ can be considered as a polynomial approximation of the function f.

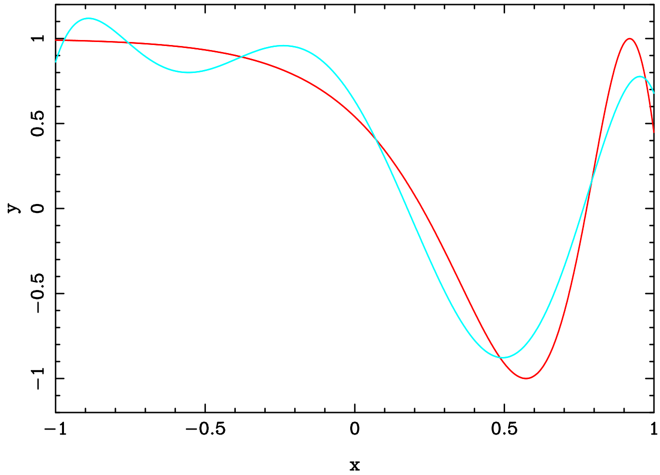
Example: Chebyshev projection of $f(x) = \cos(2\exp(x))$

$f(x) = \cos(2\exp(x))$ $\quad w(x) = (1 - x^2)^{-1/2}$ $\quad N = 4 : \|f - \Pi_4^w f\|_\infty \simeq 0.66$



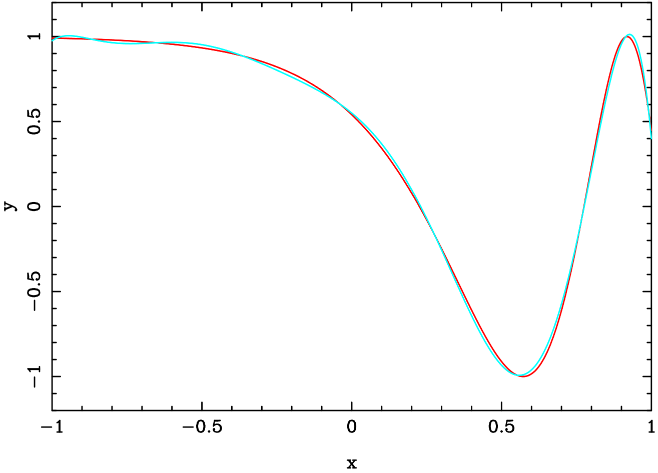
Example: Chebyshev projection of $f(x) = \cos(2\exp(x))$

$f(x) = \cos(2\exp(x))$ $\quad w(x) = (1 - x^2)^{-1/2}$ $\quad N = 6 : \|f - \Pi_6^w f\|_\infty \simeq 0.30$



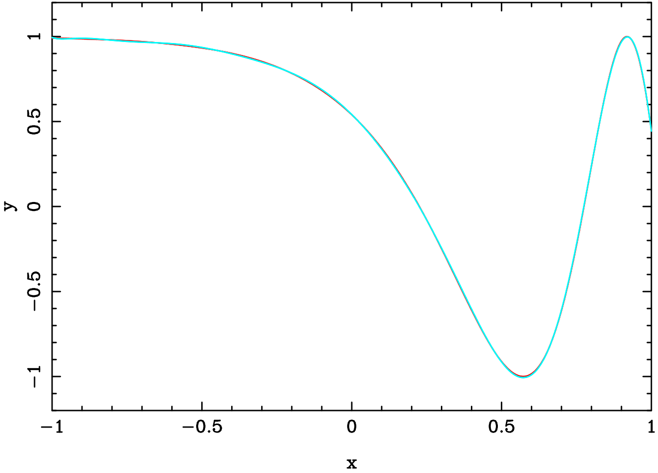
Example: Chebyshev projection of $f(x) = \cos(2\exp(x))$

$f(x) = \cos(2\exp(x))$ $\quad w(x) = (1-x^2)^{-1/2}$ $\quad N = 8 : \|f - \Pi_8^w f\|_\infty \simeq 4.9\ 10^{-2}$



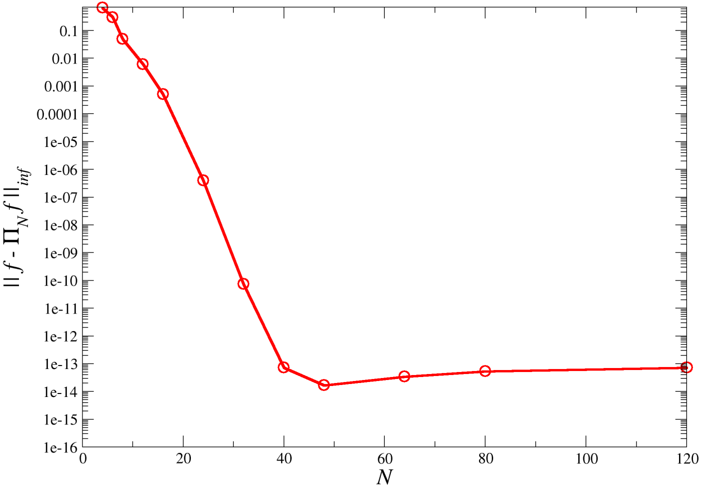
Example: Chebyshev projection of $f(x) = \cos(2\exp(x))$

$f(x) = \cos(2\exp(x))$ $w(x) = (1 - x^2)^{-1/2}$ $N = 12$: $\|f - \Pi_{12}^w f\|_\infty \simeq 6.1\ 10^{-3}$



Example: Chebyshev projection of $f(x) = \cos(2\exp(x))$

Variation of the projection error $\|f - \Pi_N^w f\|_\infty$ as N increases



Evaluation of the coefficients

The coefficients $\tilde{f}_i$ of the orthogonal projection of f are given by

$$\tilde{f}_i := \frac{(f|p_i)_w}{\|p_i\|_w^2} = \frac{1}{\|p_i\|_w^2} \int_{-1}^{1} f(x)\, p_i(x)\, w(x)\, dx \qquad (2)$$

Problem: the above integral cannot be computed exactly; we must seek a numerical approximation.

Solution: **Gaussian quadrature**

Gaussian quadrature

Theorem (Gauss, Jacobi)

Let $(p_i)_{i\in\mathbb{N}}$ be a family of orthogonal polynomials with respect to some weight w. For $N > 0$, let $X = (x_i)_{0\le i\le N}$ be the grid formed by the $N+1$ zeros of the polynomial p_{N+1} and

$$w_i := \int_{-1}^{1} \ell_i^X(x)\, w(x)\, dx$$

where ℓ_i^X is the i-th Lagrange cardinal polynomial of the grid X reminder

Then

$$\boxed{\forall f \in \mathbb{P}_{2N+1},\ \int_{-1}^{1} f(x)\, w(x)\, dx = \sum_{i=0}^{N} w_i f(x_i)}$$

If $f \notin \mathbb{P}_{2N+1}$, the above formula provides a good approximation of the integral.

Gauss-Lobatto quadrature

The nodes of the Gauss quadrature, being the zeros of p_{N+1}, do not encompass the boundaries -1 and 1 of the interval $[-1, 1]$. For numerical purpose, it is desirable to include these points in the boundaries.

This possible at the price of reducing by 2 units the degree of exactness of the Gauss quadrature

Gauss-Lobatto quadrature

Theorem (Gauss-Lobatto quadrature)

Let $(p_i)_{i\in\mathbb{N}}$ *be a family of orthogonal polynomials with respect to some weight* w*. For* $N > 0$*, let* $X = (x_i)_{0\le i\le N}$ *be the grid formed by the* $N+1$ *zeros of the polynomial*

$$q_{N+1} = p_{N+1} + \alpha p_N + \beta p_{N-1}$$

where the coefficients α *and* β *are such that* $x_0 = -1$ *and* $x_N = 1$*.*
Let

$$w_i := \int_{-1}^{1} \ell_i^X(x)\, w(x)\, dx$$

where ℓ_i^X *is the* i*-th Lagrange cardinal polynomial of the grid X.*
Then

$$\forall f \in \mathbb{P}_{2N-1}, \quad \int_{-1}^{1} f(x)\, w(x)\, dx = \sum_{i=0}^{N} w_i f(x_i)$$

Notice: $f \in \mathbb{P}_{2N-1}$ instead of $f \in \mathbb{P}_{2N+1}$ for Gauss quadrature.

Gauss-Lobatto quadrature

Remark: if the (p_i) are Jacobi polynomials, i.e. if $w(x) = (1-x)^\alpha (1+x)^\beta$, then the Gauss-Lobatto nodes which are strictly inside $(-1,1)$, i.e. $x_1, \ldots, x_{N-1}$, are the $N-1$ zeros of the polynomial p'_N, or equivalently the points where the polynomial p_N is extremal.

This of course holds for Legendre and Chebyshev polynomials.
For Chebyshev polynomials, the Gauss-Lobatto nodes and weights have simple expressions:

$$x_i = -\cos\frac{\pi i}{N}, \quad 0 \le i \le N$$

$$w_0 = w_N = \frac{\pi}{2N}, \quad w_i = \frac{\pi}{N}, \quad 1 \le i \le N-1$$

Note: in the following, we consider only Gauss-Lobatto quadratures

Discrete scalar product

The Gauss-Lobatto quadrature motivates the introduction of the following scalar product:

$$\langle f|g\rangle_N = \sum_{i=0}^{N} w_i f(x_i) g(x_i)$$

It is called the discrete scalar product associated with the Gauss-Lobatto nodes $X = (x_i)_{0\le i\le N}$

Setting $\gamma_i := \langle p_i|p_i\rangle_N$, the discrete coefficients associated with a function f are given by

$$\hat{f}_i := \frac{1}{\gamma_i}\langle f|p_i\rangle_N, \qquad 0 \le i \le N$$

which can be seen as approximate values of the coefficients $\tilde{f}_i$ provided by the Gauss-Lobatto quadrature [cf. Eq. (2)]

Discrete coefficients and interpolating polynomial

Let $I_N^{\mathrm{GL}} f$ be the interpolant of f at the Gauss-Lobatto nodes $X = (x_i)_{0 \leq i \leq N}$. Being a polynomial of degree N, it is expandable as

$$I_N^{\mathrm{GL}} f(x) = \sum_{i=0}^{N} a_i \, p_i(x)$$

Then, since $I_N^{\mathrm{GL}} f(x_j) = f(x_j)$,

$$\hat{f}_i = \frac{1}{\gamma_i} \langle f | p_i \rangle_N = \frac{1}{\gamma_i} \langle I_N^{\mathrm{GL}} f | p_i \rangle_N = \frac{1}{\gamma_i} \sum_{j=0}^{N} a_j \langle p_j | p_i \rangle_N$$

Now, if $j = i$, $\langle p_j | p_i \rangle_N = \gamma_i$ by definition. If $j \neq i$, $p_j p_i \in \mathbb{P}_{2N-1}$ so that the Gauss-Lobatto formula holds and gives $\langle p_j | p_i \rangle_N = (p_j | p_i)_w = 0$. Thus we conclude that $\langle p_j | p_i \rangle_N = \gamma_i \delta_{ij}$ so that the above equation yields $\hat{f}_i = a_i$, i.e. **the discrete coefficients are nothing but the coefficients of the expansion of the interpolant at the Gauss-Lobato nodes**

Spectral representation of a function

In a spectral method, the numerical representation of a function f is through its interpolant at the Gauss-Lobatto nodes:

$$I_N^{\mathsf{GL}} f(x) = \sum_{i=0}^{N} \hat{f}_i \, p_i(x)$$

The discrete coefficients $\hat{f}_i$ are computed as

$$\hat{f}_i = \frac{1}{\gamma_i} \sum_{j=0}^{N} w_j \, f(x_j) p_i(x_j)$$

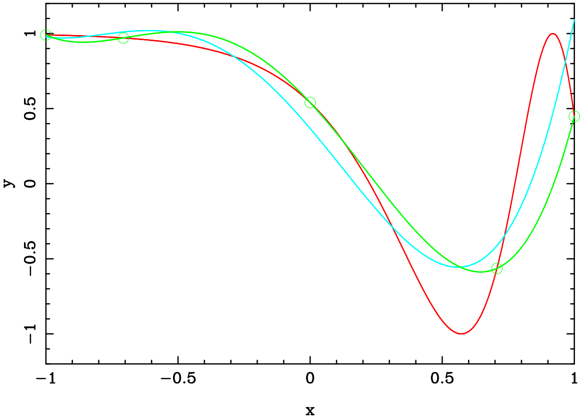
$I_N^{\mathsf{GL}} f(x)$ is an approximation of the truncated series $\Pi_N^w f(x) = \sum_{i=0}^{N} \tilde{f}_i \, p_i(x)$, which is the orthogonal projection of f onto the polynomial space $\mathbb{P}_N$.

$\Pi_N^w f$ should be the true spectral representation of f, but in general it is not computable exactly.

The difference between $I_N^{\mathsf{GL}} f$ and $\Pi_N^w f$ is called the aliasing error

Example: aliasing error for $f(x) = \cos(2\exp(x))$

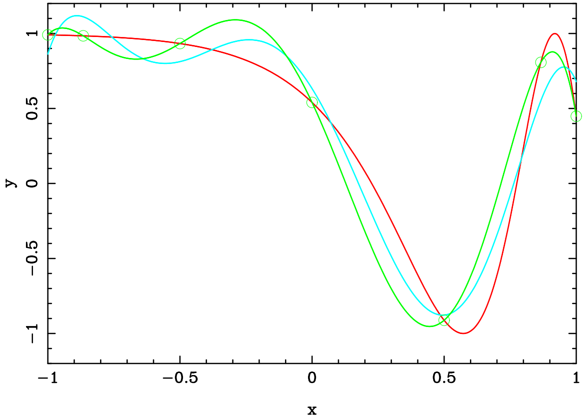
$f(x) = \cos(2\exp(x)) \quad w(x) = (1-x^2)^{-1/2} \quad N = 4$



red: f; blue: $\Pi_N^w f$; green: $I_N^{\mathrm{GL}} f$

Example: aliasing error for $f(x) = \cos(2\exp(x))$

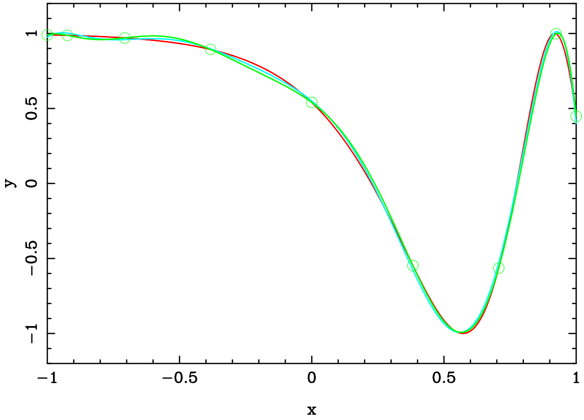
$f(x) = \cos(2\exp(x)) \quad w(x) = (1-x^2)^{-1/2} \quad N = 6$



red: f; blue: $\Pi_N^w f$; green: $I_N^{\mathrm{GL}} f$

Example: aliasing error for $f(x) = \cos(2\exp(x))$

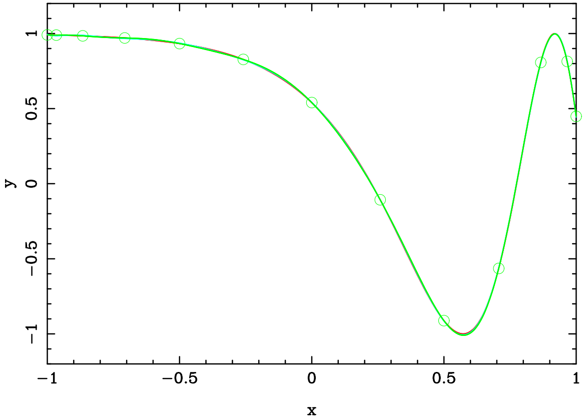
$f(x) = \cos(2\exp(x)) \qquad w(x) = (1-x^2)^{-1/2} \qquad N = 8$



red: f; blue: $\Pi_N^w f$; green: $I_N^{\mathrm{GL}} f$

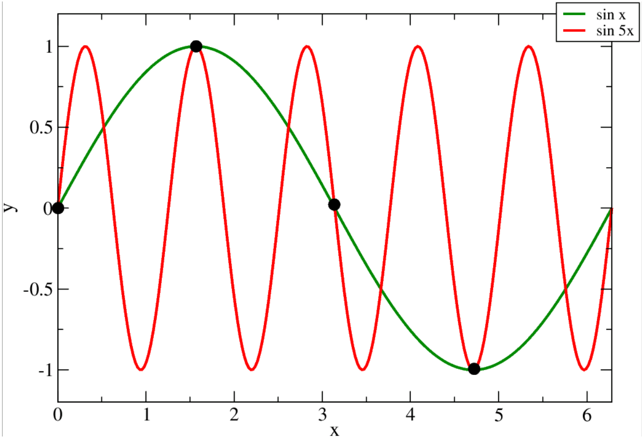
Example: aliasing error for $f(x) = \cos(2\exp(x))$

$f(x) = \cos(2\exp(x)) \quad w(x) = (1-x^2)^{-1/2} \quad N = 12$



red: f; blue: $\Pi_N^w f$; green: $I_N^{GL} f$

Aliasing error = contamination by high frequencies



Aliasing of a $\sin(x)$ wave by a $\sin(5x)$ wave on a 4-points grid

Outline

Sobolev norm

Let us consider a function $f \in C^m([-1,1])$, with $m \geq 0$.

The Sobolev norm of f with respect to some weight function w is

$$\|f\|_{H_w^m} := \left(\sum_{k=0}^{m} \left\| f^{(k)} \right\|_w^2 \right)^{1/2}$$

Convergence rates for $f \in C^m([-1,1])$

Chebyshev expansions:

- truncation error :
$$\|f - \Pi_N^w f\|_w \le \frac{C_1}{N^m} \|f\|_{H_w^m} \text{ and } \|f - \Pi_N^w f\|_\infty \le \frac{C_2(1+\ln N)}{N^m} \sum_{k=0}^{m} \left\| f^{(k)} \right\|_\infty$$
- interpolation error :
$$\left\| f - I_N^{\mathrm{GL}} f \right\|_w \le \frac{C_3}{N^m} \|f\|_{H_w^m} \text{ and } \left\| f - I_N^{\mathrm{GL}} f \right\|_\infty \le \frac{C_4}{N^{m-1/2}} \|f\|_{H_w^m}$$

Legendre expansions:

- truncation error :
$$\|f - \Pi_N^w f\|_w \le \frac{C_1}{N^m} \|f\|_{H_w^m} \text{ and } \|f - \Pi_N^w f\|_\infty \le \frac{C_2}{N^{m-1/2}} V(f^{(m)})$$
- interpolation error :
$$\left\| f - I_N^{\mathrm{GL}} f \right\|_w \le \frac{C_3}{N^{m-1/2}} \|f\|_{H_w^m}$$

Evanescent error for smooth functions

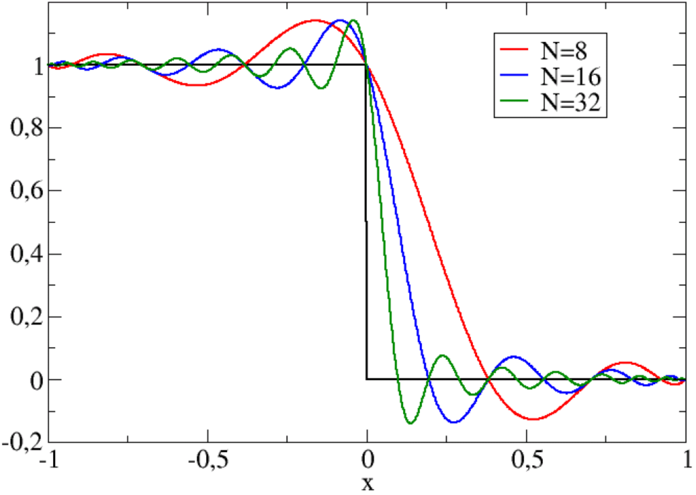
If $f \in C^\infty([-1,1])$, the error of the spectral expansions $\Pi_N^w f$ or $I_N^{\rm GL} f$ decays more rapidly than any power of N.

In practice: **exponential decay** example

This error is called evanescent.

For non-smooth functions: Gibbs phenomenon

Extreme case: f discontinuous



Outline

References

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